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Math 12 Honours Section 6.1 Imaginary and Complex Numbers

1. What is the difference between an "imaginary" number and a "complex" number? Explain:

An imaginary number has an imaginary component with "i" $= \sqrt{-1}$

2. Given a complex number: $z = 11 - 13i$, what is the value of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$?

$\mathrm{Re}(z) = 11 \quad \mathrm{Im}(z) = -13$

3. What happens whenever you multiply a complex number with it's conjugate?

The result will be a real number. The "i"s cancel out.

4. Suppose $z + \bar{z} = 10$, what do you know about the $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$?

$z = a + bi \quad z + \bar{z} = a + bi + a - bi = 10 \Rightarrow 2a = 10 \Rightarrow a = 5 \quad \mathrm{Im}(z) \in \mathbb{R}$

5. Suppose we are told that $z = \bar{z}$, what does this mean?

$z = \bar{z} \Rightarrow \cancel{a} + bi = \cancel{a} - bi \Rightarrow 2bi = 0 \Rightarrow b = 0$ No imaginary component! $\mathrm{Im}(z) = 0$

6. What does $|z|$ represent? What does it mean? Explain?

$|z| = \sqrt{a^2 + b^2} = r$

7. Given that $z_1 \times z_2 = 7 + 8i$, then what is the value of $\bar{z_1} \times \bar{z_2} = ?$

$\overline{z_1} \times \overline{z_2} = \overline{z_1 \times z_2} = \overline{7 + 8i} = \boxed{7 - 8i}$

8. Given that $z_1 + z_2 = 55 - 45i$, then what is the value of $\bar{z_1} + \bar{z_2}$?

$\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2} = \boxed{55 + 45i}$

9. Solve for "x" and present your solution in the form of $a \pm bi$

$\frac{42}{4}$ $\frac{160}{8}$

a) $3x^2 - 2x + 7 = 0$ $ax^2 + bx + c = 0$ $\quad a = 3,\ b = -2,\ c = 7$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{2 \pm \sqrt{4 - 84}}{6} = \frac{2 \pm 4i\sqrt{5}}{6}$ $= \boxed{\frac{1}{3} \pm \frac{2\sqrt{5}}{3} i}$	b) $(x^2 + 9)(x^2 + 100) = 0$ $x^2 + 9 = 0 \quad x^2 + 100 = 0$ $\boxed{x = \pm 3i} \quad \boxed{x = \pm 10i}$	c) $7x^2 - 5x + 6 = 0$ $x = \frac{5 \pm \sqrt{25 - 4 \cdot 7 \cdot 6}}{14} = \boxed{\frac{5 \pm i\sqrt{143}}{14}}$
d) $-2(x+6)^2 + 1 = 65$ $(x+6)^2 = -32$ $x + 6 = \pm 4i\sqrt{2}$ $\boxed{x = -6 \pm 4i\sqrt{2}}$	e) $4(x+3)^2 + 25 = 0$ $(x+3)^2 = -\frac{25}{4}$ $x + 3 = \pm \frac{5i}{2}$ $\boxed{x = -3 \pm \frac{5}{2} i}$	f) $x^4 + 16x^2 = 225$ $x^2 = \frac{-16 \pm \sqrt{16^2 - 4(-225)}}{2} = \frac{-16 \pm 34}{2}$ $x^2 = -25 \Rightarrow \boxed{x = \pm 5i}$ $x^2 = 9 \Rightarrow \boxed{x = \pm 3}$

g) $x^2 - \left(\frac{2}{x}\right)^2 = 3$

$x^2 - \frac{4}{x^2} = 3$

$x^4 - 4 = 3x^2$

$x^4 - 3x^2 - 4 = 0$

$(x^2-4)(x^2+1)=0$

$\boxed{\begin{array}{l} x=\pm 2 \\ x=\pm i \end{array}}$

h) $\frac{15}{x+3} - \frac{x}{x-3} = 1$

$\frac{15(x-3) - x(x+3)}{x^2-9} = 1$

$15x - 45 - x^2 - 3x = x^2 - 9$

$2x^2 - 12x + 36 = 0$

$x^2 - 6x + 18 = 0$

$x = \frac{6 \pm \sqrt{36-72}}{2} \Rightarrow \boxed{x = 3 \pm 3i}$

i) $\frac{2}{x+5} - \frac{x}{x-5} = 5$

$2x - 10 - x^2 - 5x = 5x^2 - 125$

$6x^2 - 3x - 115 = 0$

$x = \frac{3 \pm \sqrt{9 - 4(6)(-115)}}{12} = \boxed{\frac{3\pm\sqrt{2769}}{12}}$

10. Simplify or evaluate the following and express your answer in the form of $a \pm bi$:

a) $(3+2i)(1-3i)$

$3-7i+6 = \boxed{9-7i}$

b) $\left(2-\sqrt{-4}\right)+\left(-3+\sqrt{-16}\right)$

$2-3+4i-2i = \boxed{-1+2i}$

c) $(-1+i)(i+1)+(3+i)(3-i)$

$Z \times \bar{Z}$

$= i^2-1^2 = -2$ $\quad = 9-i^2 = 10$

$= \boxed{8}$

d) $\frac{1+i}{1-i} - \frac{1-i}{1+i}$

$\frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)} = \frac{\cancel{1}+2i\cancel{-1}\cancel{-1}+2i\cancel{+1}}{1+1}$

$= \frac{4i}{2} = \boxed{2i}$

e) $\sqrt{\frac{-3}{2}} + \sqrt{\frac{-2}{3}}$

$\frac{i\sqrt{6}}{2} + \frac{i\sqrt{6}}{3} = \boxed{\frac{5i\sqrt{6}}{6}}$

f) $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$

$\frac{(1+2i)5i + (2-i)(3-4i)}{(3-4i)5i}$

$= \frac{5i-10+6-11i-4}{15i+20}$

$= \frac{-6i-8}{15i+20} = \frac{-2\cancel{(3i+4)}}{5\cancel{(3i+4)}} = \boxed{-\frac{2}{5}}$

g) $\frac{1+2i}{3+4i} + \frac{2i-5}{5i}$

$= \frac{(1+2i)5i+(2i-5)(3+4i)}{5i(3+4i)}$

$= \frac{5i-10+6i-8-15-20i}{15i-20}$

$= \frac{-9i-33}{15i-20} \times \frac{15i+20}{15i+20} = \frac{135-180i-495i-660}{-225-400}$

$= \frac{675i-525}{-625} = \frac{135i+105}{-125} = \boxed{\frac{27i-21}{-25}}$

h) $\left(\sqrt{9+40i}+\sqrt{9-40i}\right)^2$

$= 9+\cancel{40i}+2\sqrt{81+1600}+9-\cancel{40i}$

$= 18+2\cdot 41 = \boxed{100}$

i) $\frac{(2+i)^2}{2-i} + \frac{(2-i)^2}{2+i}$

$= \frac{(2+i)(3+4i)+(2-i)(3-4i)}{4+1}$

$= \frac{6+\cancel{11i}-4+6-\cancel{11i}-4}{5}$

$= \boxed{\frac{4}{5}}$

11. Find the values of "a" and "b":

a) $a+ib=\sqrt{153+104i}$

$a^2+2abi-b^2=153+104i$

$a^2-b^2=153 \qquad 2ab=104$

$ab=52$

$\frac{52^2}{b^2}-b^2=153 \qquad a=\frac{52}{b}$

$52^2-b^4=153b^2$

$b^4+153b^2-52^2=0$

$b^2=\frac{-153\pm\sqrt{153^2+4(52)^2}}{2}$

$b^2=\frac{-153\pm185}{2}=16,-169$

$\boxed{(a,b)=(13,4),(-13,-4)}$

b) $a+ib=\sqrt{-16-30i}$

$a^2-b^2=-16 \qquad 2ab=-30$

$ab=-15$

$\left(\frac{-15}{b}\right)^2-b^2=-16 \qquad a=\frac{-15}{b}$

$225-b^4=-16b^2$

$b^4-16b^2-225=0$

$b^2=\frac{16\pm\sqrt{16^2+4\cdot225}}{2}$

$b^2=\frac{16\pm34}{2}=25,-9$

$\boxed{(a,b)=(3,-5),(-3,5)}$

c) $a+ib=\sqrt{-15+112i}$

$a^2-b^2=-15 \qquad 2ab=112$

$ab=56$

$\frac{56^2}{b^2}-b^2=-15 \qquad a=\frac{56}{b}$

$56^2-b^4=-15b^2$

$b^4-15b^2-56^2=0$

$b^2=\frac{15\pm\sqrt{15^2+4\cdot56^2}}{2}$

$b^2=\frac{15\pm113}{2}=64,-49$

$\boxed{(a,b)=(7,8),(-7,-8)}$

12. Given that "z" is a complex number in the form of $a\pm bi$, solve for "z"

a) $5z^2+4=0$

$5z^2=-4$

$z^2=-\frac{4}{5}$

$z=\pm\sqrt{-\frac{4}{5}}$

$\boxed{z=\frac{2\sqrt{5}i}{5},\ \frac{-2\sqrt{5}i}{5}}$

b) $z^2=5-12i$

$(a+bi)^2=5-12i$

$a^2-b^2=5 \qquad 2ab=-12$

$ab=-6$

$\frac{36}{b^2}-b^2=5 \qquad a=\frac{-6}{b}$

$36-b^4=5b^2$

$b^4+5b^2-36=0$

$(b^2-4)(b^2+9)=0$

$b=\pm2 \qquad b=\pm3i$

$(a,b)=(3,-2),(-3,2)$

$\boxed{z=3-2i,\ -3+2i}$

c) $z^2=-3+4i$

$z^2=a^2+2abi-b^2=-3+4i$

$a^2-b^2=-3 \qquad ab=2$

$\frac{4}{b^2}-b^2=-3 \qquad a=\frac{2}{b}$

$4-b^4=-3b^2$

$b^4-3b^2-4=0$

$(b^2-4)(b^2+1)=0$

$b^2=4,-1$

$(a,b)=(1,2),(-1,-2)$

$\boxed{z=1+2i,\ -1-2i}$

d) $z^2+(i-5)z+12-5i=0$

$z=\frac{5-i\pm\sqrt{(i-5)^2-4(12-5i)}}{2}$

$z=\frac{5-i\pm\sqrt{24-10i-48+20i}}{2}$

$z=\frac{5-i\pm\sqrt{10i-24}}{2}$

$z=\frac{5-i\pm(1+5i)}{2}$

$\boxed{z_1=3+2i \quad z_2=2-3i}$

e) $(5-2i)-(z+4i)=7-6i$

$5-2i-z-4i=7-6i$

$\boxed{z=-2}$

f) $z^3=8$

$z_1=2$

$z_2=2e^{120i}=-1+\sqrt{3}i$

$z_3=2e^{-120i}=-1-\sqrt{3}i$

g) $z^2 - 15 + 8i = 0$

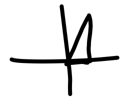
$z^2 = 15 - 8i$

$z^2 = 17e^{-28.1^\circ i}$

$z_1 = \sqrt{17}e^{-14.04^\circ i} = 4 - i$

$z_2 = \sqrt{17}e^{165.96^\circ i} = -4 + i$

g) $z^2 - 16 - 16i\sqrt{3} = 0$



$z^2 = 16 + 16\sqrt{3}i$

$z^2 = 16(1 + \sqrt{3}i)$

$z^2 = 16(2e^{60i})$

$z^2 = 32e^{60i}$

$z_1 = 4\sqrt{2}e^{30i} = \boxed{2\sqrt{6} + 2\sqrt{2}i}$

$z_2 = 4\sqrt{2}e^{-150i} = \boxed{-2\sqrt{6} - 2\sqrt{2}i}$

h) $z - \sqrt{-144} - \left(3\sqrt{-i} + 1\right)^2 = 7 - 6i$

$z = 7 - 6i + (3\sqrt{-i} + 1)^2 + \sqrt{-144}$

$z = 7 - 6i - 9i + 6\sqrt{-i} + 1 + 12i$

$z = 8 - 3i + 6\left(\pm\frac{1 - i}{\sqrt{2}}\right)$

$\boxed{z_1 = 8 + 3\sqrt{2} - 3i + 3\sqrt{2}i}$

$\boxed{z_2 = 8 - 3\sqrt{2} - 3i - 3\sqrt{2}i}$

13. Evaluate $i^{2021} \times i^{2020} \times i^{2019} \times i^{2018}$

$i^{2018} = (i^2)^{1009} = (-1)^{1009} = -1$

$= -1 \cdot -i \cdot 1 \cdot i = \boxed{-1}$

14. Find the value of $(-i)^{4n-1}$ when "n" is a negative odd integer.

let $n = -2k + 1$

$(-i)^{4(-2k+1)-1} = (-i)^{-8k+3} = (-i)^{-8k} \cdot (-i)^3 = \frac{1}{(-i^2)^{4k}} \cdot i = 1 \cdot i = \boxed{i}$

15. If "z" is a complex number and $\bar{z}$ is its conjugate, then determine the complex numbers which satisfy the equation: $5z^2 - 4z(\bar{z}) = (1 - 3i)z$

let $z = a + bi$

$5(a+bi)^2 - 4(a+bi)(a-bi) = (1-3i)(a+bi)$

$5a^2 + 10abi - 5b^2 - 4a^2 - 4b^2 = a + bi - 3ai + 3b$

$\underbrace{a^2 - 9b^2}_{\text{real}} + \underbrace{10abi}_{\text{imaginary}} = \underbrace{a + 3b}_{\text{real}} + \underbrace{i(b - 3a)}_{\text{imaginary}}$

$a^2 - 9b^2 = a + 3b$

$10abi = (b - 3a)i$

$\Rightarrow (a, b) = (0, 0), (1, -\frac{1}{3})$

$\boxed{z = 0,\ 1 - \frac{1}{3}i}$

16. Given that $f(x) = (-2 + i)x^2 - (3 + i)x + 4 - 5i$, find the value for each of the following:

i) $f(i)$

$f(i) = (-2+i)(-1) - (3+i)i + 4 - 5i$

$= 2 - i - 3i + 1 + 4 - 5i$

$= \boxed{7 - 9i}$

ii) $f(1+i)$

$f(1+i) = (-2+i)(1+i)^2 - (3+i)(1+i) + 4 - 5i$

$= (-2+i)(\cancel{1} + 2i \cancel{-1}) - (3 + 4i - 1) + 4 - 5i$

$= -4i - \cancel{2} - \cancel{2} - 4i + \cancel{4} - 5i$

$= \boxed{-13i}$

iii) $f(3-i)$

$= (-2+i)(3-i)^2 - (3+i)(3-i) + 4 - 5i$

$= (-2+i)(9 - 6i - 1) - (9 + 1) + 4 - 5i$

$= -16 + 12i + 8i + 6 - 10 + 4 - 5i$

$= -10 + 20i - 10 + 4 - 5i$

$= \boxed{-16 + 15i}$

17. If "z" is a complex number and $\bar{z}$ is its conjugate, then determine the value of : $z^5 - \left(\bar{z}\right)^5$

$Z = a+bi$

$z^5 = (a+bi)(a+bi)(a+bi)(a+bi)(a+bi)$

The only real terms in z^5 have 0, 2, or 4 "bi"s.

$\therefore \operatorname{Re}(z^5) = a^5 - {}_5C_2\, a^3b^2 + {}_5C_1\, ab^4$

$(\bar{z})^5 = \overline{z^5} =$

$z^5 + \overline{z^5} = 2\operatorname{Re}(z^5) = 2(a^5 + 10a^3b^2 + 5ab^4) = \boxed{2a^5 + 20a^3b^2 + 10ab^4}$

18. Find the sum of the following: $1 + 2i + 3i^2 + 4i^3 + ... + 1000i^{999} + 1001i^{1000}$

$i = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$= \underbrace{1 + 2i - 3 - 4i}_{= -2-2i} + \underbrace{5 + 6i - 7 - 8i}_{= -2-2i} + \underbrace{9 + 10i - 11 - 12i}_{= -2-2i} + \cdots + 1001$

$= (-2-2i)250 + 1001 = \boxed{501 - 500i}$

19. There is a complex number "z" with imaginary part 164 and a positive integer "n" such that: $\dfrac{z}{z+n} = 4i$.

Find the value of "n". AIME I 2009 $Z = a + 164i$

$Z = 4iz + 4in$

$z(1-4i) = 4in$

$z = \dfrac{4in}{1-4i} \Rightarrow z = \dfrac{4in(1+4i)}{1+16} = \dfrac{4in - 16n}{17} \Rightarrow \dfrac{4n}{17} = 164 \Rightarrow n = \dfrac{17}{4} \cdot 164 = \boxed{697}$

20. Find "c" if "a", "b", and "c" are positive integers that satisfy the following equation: $c = (a+ib)^3 - 107i$

1985 AIME

$(a^2 + 2abi - b^2)(a + ib)$

$a^3 + 2a^2bi - ab^2 + a^2bi - 2ab^2 - b^3i$

$a^3 + 3a^2bi - 3ab^2 - b^3i$

$c = a^3 + 3a^2bi - 3ab^2 - b^3i - 107i$

$c + 107i = a^3 - 3ab^2 + (3a^2b - b^3)i \Longrightarrow 107 = 3a^2b - b^3$

$107 = b(3a^2 - b^2)$

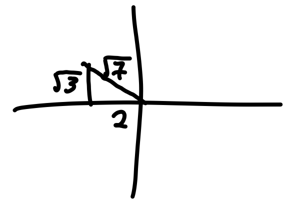
$\cancel{\begin{matrix} 1 \\ 107 \end{matrix}} \quad \begin{matrix} 107 \\ 1 \end{matrix} \longrightarrow b = 1,\ a = 6$

$c = a^3 - 3ab^2 = 6^3 - 3(6)(1)^2 = \boxed{198}$

21. Given the following equation, find the value of “k” if “k” and “m” are integers:

$$\left[2-\left(-2+i\sqrt{3}\right)-\left(-2-i\sqrt{3}\right)\right]\left[2+\left(-2+i\sqrt{3}\right)^2+\left(-2-i\sqrt{3}\right)^2\right]\left[2-\left(-2+i\sqrt{3}\right)^4-\left(-2-i\sqrt{3}\right)^4\right]=2^k3^m$$

$$(2-z-\bar{z})(2+z^2+\bar{z}^2)(2-z^4-\bar{z}^4) = 2^k 3^m$$



$$(2+2-i\sqrt{3}+2+i\sqrt{3})(2+4+4-3-3)(2-49e^{163.6i}-49e^{-163.6i})$$

$$(6)(6)(2-\cancel{47}-13.\cancel{886}+\cancel{47}+13.\cancel{856}) = 36\cdot 2 = 2^3\cdot 3^2$$ $\boxed{k=3,\ m=2}$

21. Find the number of ordered pairs of real number (a,b) such that $(a+ib)^{2002}=a-bi$ (AMC 12)

a) 1001 b) 1002 c) 2001 d) 2002 e) 2004

$$(a+ib)^{2002} = a-bi$$

OR, $r=0$

1 solution

$$(re^{i\theta})^{2002} = re^{-i\theta} \Rightarrow r=1$$

$$z^{2002} = \frac{\cancel{r^2}\,1}{\bar{z}}$$

$$z^{2002} = \frac{1}{z} \Rightarrow z^{2003} = 1$$ By roots of unity, we have 2003 solutions here

2003 + 1 = 2004 solutions